An Option Value Problem from *Seinfeld*[*]

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Abstract

This is a paper about nothing.

1 Introduction

In an episode of the sitcom *Seinfeld,* Elaine Benes uses a contraceptive sponge that gets taken off the market. She scours pharmacies in the neighborhood to stock a large supply, but it is finite. So she must "re-evaluate her whole screening process." Every time she dates a new man, which happens very frequently, she has to consider a new issue: Is he "spongeworthy"? The purpose of this article is to quantify this concept of spongeworthiness.

When Elaine uses up a sponge, she is giving up the option to have it available when an even better man comes along. Therefore using the sponge amounts to exercising a real option to wait, and spongeworthiness is an option value. It can be calculated using standard option-pricing techniques. However, unlike the standard theory of financial or many real options, there are no complete markets and no replicating portfolios. Stochastic dynamic programming methods must be used.

2 The Model

Suppose Elaine believes herself to be infinitely lived; this is a good approximation in relation to the number of sponges she has and her time-discount factor or impatience. She meets a new man every day. Define the quality Q of each man as Elaine's utility from having sex with him. This is independently and identically distributed, and drawn each day from a distribution which I assume to be uniform over [0,1]. Each day she observes the Q of that day's date. Actually this is only her estimate formed from observing and closely questioning

[*] I developed this model many years ago, but kept it hidden because it seemed too politically incorrect. I hope that my advanced age now exempts me from the constraints of political correctness.

I thank Ricardo Guzmán for his abstract suggestion.

the man (which is what she does in the episode), not the ex post facto outcome. But I assume that she has sufficient experience and expertise to make a very accurate estimate. Having observed Q, she makes her yes/no decision. Elaine's per-day discount factor is β. All these assumptions are to simplify the calculations; the method is perfectly general and many bells and whistles can be added to the analysis. I mention a few of them at the end.

If sponges were freely available for purchase at a constant price (which is small in relation to the potential value so I will ignore it), then Elaine's decision would be yes for any quality greater than zero. But when she has a finite stock and cannot buy any more, her optimal decision will be based on a "spongeworthiness threshold" of quality, Q_m, such that her decision will be yes if $Q > Q_m$. The threshold depends on the number m of sponges she has: the fewer sponges left, the higher the threshold needed to justify using up one of them.

Let V_m denote Elaine's expected present value of utility when she has a stock of m sponges. She meets a man and observes his quality Q. If she decides to use one of her sponges, she gets the immediate payoff Q and has continuation value V_{m-1} on the second day,which has present value $\beta\, V_{m-1}$. If she decides not to, there is no immediate payoff, only the present value of continuation with m sponges, namely $\beta\, V_m$. Therefore her decision rule is

$$\begin{aligned} \text{Spongeworthy if} \qquad & Q + \beta\, V_{m-1} > \beta\, V_m \\ \text{that is, if} \qquad & Q > Q_m \equiv \beta\,(V_m - V_{m-1})\,, \end{aligned} \tag{1}$$

and the value recursion formula of dynamic programming is

$$V_m = \mathbf{E}\,[\ \max\{\, Q + \beta\, V_{m-1}\,,\ \beta\, V_m\,\}\ ]\ . \tag{2}$$

Therefore

$$\begin{aligned} V_m &= \int_0^{Q_m} \beta\, V_m\, dq + \int_{Q_m}^1 (q + \beta\, V_{m-1})\, dq \\ &= \beta\, V_m\, Q_m + \tfrac{1}{2}\,\left[\, 1 - (Q_m)^2\,\right] + \beta\, V_{m-1}\,(1 - Q_m) \\ &= \beta\,(V_m - V_{m-1})\; Q_m + \tfrac{1}{2} - \tfrac{1}{2}\,(Q_m)^2 + \beta\, V_{m-1} \\ &= (Q_m)^2 + \tfrac{1}{2} - \tfrac{1}{2}\,(Q_m)^2 + \beta\, V_{m-1} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\,(Q_m)^2 + \beta\, V_{m-1} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\,\beta^2\,(\,V_m - V_{m-1}\,)^2 + \beta\, V_{m-1}\,. \end{aligned}$$

Write this as

$$V_m - V_{m-1} = \tfrac{1}{2} + \tfrac{1}{2}\,\beta^2\,(\,V_m - V_{m-1}\,)^2 - (1-\beta)\, V_{m-1}\,,$$

or

$$\beta^2\,(\,V_m - V_{m-1}\,)^2 - 2\,(\,V_m - V_{m-1}\,) + [1 - 2(1-\beta)\, V_{m-1}] = 0\,.$$

Therefore

$$\begin{aligned} V_m - V_{m-1} &= \frac{2 \pm \sqrt{4 - 4\,\beta^2\,[1 - 2(1-\beta)\, V_{m-1}]}}{4\,\beta^2} \\ &= \frac{1 \pm \sqrt{1 - \beta^2 + 2\,\beta^2\,(1-\beta)\, V_{m-1}}}{\beta^2}\,. \end{aligned}$$

The initial condition is $V_0 = 0$. Keeping the positive root would make $V_1 > 1/\beta^2 > 1$ which is impossible. Therefore keep the negative root and write the difference equation as

$$V_m = V_{m-1} + \frac{1 - \sqrt{1 - \beta^2 + 2\,\beta^2\,(1-\beta)\,V_{m-1}}}{\beta^2}\,. \tag{3}$$

3 Solution

To solve this, examine the function

$$f(x) = x + \frac{1 - \sqrt{1 - \beta^2 + 2\,\beta^2\,(1-\beta)\,x}}{\beta^2}\,. \tag{4}$$

We have

$$f(0) = \frac{1 - \sqrt{1-\beta^2}}{\beta^2} > 0\,,$$

and for large x, the second term (built-up fraction) on the right hand side of (4) becomes negative so eventually $f(x) < x$. Also

$$\begin{aligned} f'(x) &= 1 - \frac{1}{\beta^2}\,\tfrac{1}{2}\,\Big(1 - \beta^2 + 2\,\beta^2\,(1-\beta)\,x\Big)^{-1/2}\;2\,\beta^2\,(1-\beta) \\ &= 1 - (1-\beta)\;\Big(1 - \beta^2 + 2\,\beta^2\,(1-\beta)\,x\Big)^{-1/2}\,. \end{aligned}$$

This is increasing as x increases. At the extremes,

$$f'(0) = 1 - (1-\beta)\;\Big(1-\beta^2\Big)^{-1/2} = 1 - \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\,,$$

which is positive but less than one, and

$$f'(\infty) = 1\,.$$

Figure 1 shows this curve. Starting at $x = V_0 = 0$, we successively read off $V_1 = f(V_0)$, $V_2 = f(V_1)$ etc. As $m \to \infty$, $V_m \to V^*$, the fixed point $f(x) = x$. Solving (4) we find

$$V^* = \frac{1}{2\,(1-\beta)}\,. \tag{5}$$

This is obviously correct: with infinitely many spongers Elaine can use one every day to have an expected value $\frac{1}{2}$ each day, and V^* is just the capitalized value of this.

Table 1 shows some numerical calculations for the values for the cases where Elaine has just one sponge left,

$$V_1 = \frac{1 - \sqrt{1-\beta^2}}{\beta^2}\,, \tag{6}$$

where she has 10 sponges left (V_{10}), and 100 left (V_{100}) (there are no simple explicit formulas for the latter two), and the implied spongeworthiness thresholds for each case, for various discount factors. Note that β is the daily discount factor; therefore for better intuition I also show the annual discount factor β^{365} in each case.

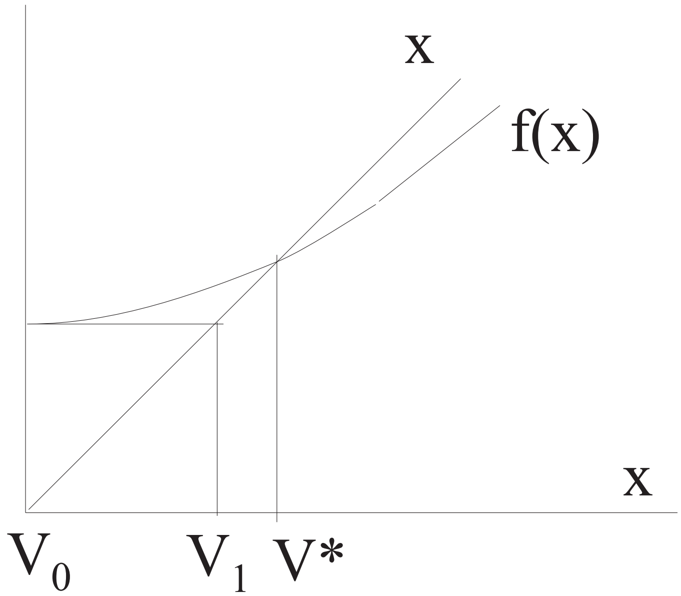

Figure 1: Graphical solution

Table 1: Numerical calculations

β	β^{365}	V_1	Q_1	V_{10}	Q_{10}	V_{100}	Q_{100}
0.999999	0.999635	0.998	0.998	9.968	0.996	99.054	0.986
0.999	0.694	0.957	0.957	9.044	0.866	73.333	0.617
0.990	0.026	0.876	0.868	7.272	0.619	35.416	0.168
0.900	1.988×10^{-17}	0.696	0.627	3.571	0.155	4.999	1.73×10^{-5}
0.500	0	0.536	0.268	0.999	4.27×10^{-4}	1.000	0.000

Some limiting cases should be noted: [1] Elaine very patient: As $\beta \to 1$, from (3) we have $V_m - V_{m-1} \to 1$. Then $V_m \to m$. Therefore (1) gives $Q_m \to 1$: a completely patient Elaine will accept no one except the best possible man every day. The first line of the table is a close approximation to this. [2] Elaine very impatient: As $\beta \to 0$, expanding the square root in (6) in its binomial series, we have $V_1 \to 1/2$ and then $Q_1 \to 0$. [3] Large stock: For the low values of β, namely 0.9 and 0.5, where events one year out are negligible, V_{100} is a very close approximation to the V^* in (5).

4 Some Extensions

[1] Elaine has a finite life of n days, where $n > m$ the number of sponges. Defining the value function $V(m,n)$ in the obvious way, the Bellman equation is

$$V(m,n) = \mathbf{E}\,[\ \max\{\, Q + \beta\, V(m-1, n-1)\,,\ \beta\, V(m, n-1)\,\}\ ]\ .$$

This yields the threshold

$$Q(m,n) = \beta\,[\,V(m,n-1) - V(m-1,n-1)\,]\,.$$

The boundary conditions are $V(0,n) = V(m,0) = 0$ for all m, n. This can be solved numerically quite easily at least for small m, n.

[2] In the episode, when George comes asking Elaine for one of her sponges, she should be willing to sell it for the price $V(m) - V(m-1)$ (or $V(m,n) - V(m-1,n)$ in the finite life case).

[3] At the end of the episode, when the man she has accepted for the night asks for a second helping, she says "Sorry, I can't afford two of them." This pins down the man's spongeworthiness quite precisely: greater than Q_m but less than Q_{m-1}. For example, if $m = 10$ and $\beta = 0.999$, the interval is [0.866,0.872].